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GENERALIZED BARONTI CONSTANT AND NORMAL STRUCTURE

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Abstract

We introduce the definition of generalized Baronti constant $A_2(a, X)$ in Banach space, analyze some properties of this modulus, and construct the relation between this modulus and other geometrical constants, the main result is: If $A_2(a, X) < 1 + a$, for some $a \in [0, 1]$, then Banach space X has uniform normal structure.

1. Introduction

The properties which can imply metric fixed point theory in a Banach space have been studied widely. Some properties of Jordan-von Neumann constant and James constant have been shown to imply uniform normal structure [2], [4].

Baronti et al. [1] defined parameter $A_2(X)$ to inscribe normal structure. In this paper, we consider the generalized constants $A_2(a, X)$

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and give some properties. And discuss the relations through these constants. We show that, if $A_2(a, X) < (1 + a)$ for some $a \in [0, 1]$, then X possesses uniform normal structure. As an example, we compute $A_2(a, X)$, for all $a \in [0, 2]$, when X is a Hilbert space.

2. Preliminaries

Throughout the paper, we let X stand for a Banach space. By B_X and S_X, we denote the closed unit ball and the unit sphere of X, respectively. We shall say that a nonempty weakly compact convex subset C of X has the fixed point property (fpp for short), if every nonexpansive mapping $T : C \to C$ has a fixed point (that is, there exits $x \in C$ such that $T(x) = x$). Recall that T is nonexpansive, if $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in C$. We shall say that X has the fixed point property (fpp), if every weak compact convex subset of X has the fpp. Let A be a nonempty bounded set in X. The number $r(A) = \inf\{\sup_{y \in A} \|x - y\| : x \in A\}$ is called the *Chebyshev radius* of A. The number $\operatorname{diam} A = \sup\{\|x - y\| : x, y \in A\}$ is called the *diameter* of A. A Banach space X has normal structure, if

$$r(A) < \operatorname{diam} A, \tag{2.1}$$

for every bounded convex closed subset A of X with $\operatorname{diam} A > 0$. When (1.1) holds for every weakly compact convex subset A of X with $\operatorname{diam} A > 0$, we say X has weak normal structure. Normal structure and weak normal structure coincide, if X is reflexive. A space X is said to have *uniform normal structure*, if $\inf\{(\operatorname{diam} A) / (r(A))\} > 1$, where the infimum is taken over all bounded convex closed subsets A of X with $\operatorname{diam} A > 0$. Weak structure, as well as many other properties imply the fpp. The relevant papers are [8], [9], [10], and so on.

The modulus of convexity of X is a function $\delta_X : [0, 2] \to [0, 2]$ defined by $\delta_X(\varepsilon) = \inf\{1 - \frac{\|x + y\|}{2} : x, y \in S_x, \|x - y\| \geq \varepsilon\}$. If $\delta_X(1) \geq 0$, then X has uniform normal structure [2], [5].

Definition 2.1 [10]**.** Let $\mathcal{U}$ be a filter on I, then $\{x_i\}$ is said to convergence x with respect to $\mathcal{U}$, denoted by $\lim_{\mathcal{U}} x_i = x$, if for each neighborhood V of x, $\{i \in I : x_i \in V\} \in \mathcal{U}$. $\mathcal{U}$ is called an ultrafilter, if it is maximal with respect to the ordering of set inclusion. An ultrafilter is called trivial, if it is of the form $\{A : A \subseteq I, i_0 \in A\}$ for some $i_0 \in I$.

We will use the fact that: if $\mathcal{U}$ is an ultrafilter, then

(i) for any $A \subseteq I$, either $A \in \mathcal{U}$, or $I \setminus A \in \mathcal{U}$;

(ii) if $\{x_i\}$ has a cluster point x, then $\lim_{\mathcal{U}} x_i = x$.

Let $\{X_i\}$ be a family of Banach spaces, and let $l_\infty(I, X_i)$ denote the subspace of the product space $\prod X_i$, equipped with the norm $\|(x_i)\| = \sup_{i \in I}\|x_i\| < \infty$. Let $\mathcal{U}$ be an ultrafilter on I $N_{\mathcal{U}} = \{(x_i) \in l_\infty(I, X_i) : \lim_{\mathcal{U}}\|(x_i)\| = 0\}$. The ultraproduct of $\{X_i\}_{i \in I}$ is the quotient space $l_\infty(I, X_i)/N_{\mathcal{U}}$. We will use $\tilde{x}$ to denote the element of the ultraproduct. It follows from property (ii) above and the definition of quotient norm that,

$$\|\tilde{x}\| = \lim_{\mathcal{U}}\|x_i\|.$$

In what follows, we will restrict our set I to be $\boldsymbol{N}$, and let $X_i = X$, $i \in \boldsymbol{N}$, for some Banach spaces X. For an ultrafilter $\mathcal{U}$ on $\boldsymbol{N}$, we use $\tilde{X}$ denote the ultraproduct. It is also clear that X is isometric to subspace of $\tilde{X}$. Hence, we may assume that X is a subspace of $\tilde{X}$.

Lemma 2.1. *If X is a Banach space, then $(\tilde{X})^* = (\tilde{X}^*)$, iff X is super-reflexive.*

3. Main Results

Definition 3.2. Let X be a Banach space, for $a \geq 0$,

$$A_2(a, X) = \sup\left\{\frac{\|x + y\| + \|x - z\|}{2} : \|y - z\| \leq a\|x\|,\ x,\ y,\ z \in B_X\right\}$$

$$= \sup\left\{\frac{\|x+y\|+\|x-z\|}{2} : \|y-z\| \le a\|x\|,\ x,\ y,\ z \in B_X\right.$$

of which at least one belongs to $\left. S_X \right\}.$

First let us show some clear properties of $A_2(a, X)$:

(1) $A_2(0, X) = A_2(x)$;

(2) $A_2(a, X)$ is a nondecreasing function with respect to a;

(3) $1 + \frac{a}{2} \le A_2(a, X) \le 2,\ a \in [0, 2]$;

(4) If $A_2(a, X) < 2$, for some $a \ge 0$, then $A_2(X) < 2$ and consequently, X is uniformly nonsquare.

Theorem 3.2. *For a Hilbert space* H, $A_2(a, H) = \sqrt{2+a}$.

Proof. Let $x, y, z \in B_H$ with $\|y - z\| \le a\|x\|$. On one hand, we have

$$\frac{\|x+y\|+\|x-z\|}{2} \le \sqrt{\frac{\|x+y\|^2+\|x-z\|^2}{2}}$$

$$= \sqrt{\frac{2\|x\|^2+\|y\|^2+\|z\|^2+2 < x,\ y-z >}{2}}$$

$$\le \sqrt{\frac{4+2\|x\|\,\|y-z\|}{2}}$$

$$\le \sqrt{2+a}.$$

On the other hand, let e_1 and e_2 be orthonormal elements of S_H. Put

$$x = e_1,\ y = \frac{a}{2}e_1 + \sqrt{1-\frac{a^2}{4}}e_2,\ z = -\frac{a}{2}e_1 + \sqrt{1-\frac{a^2}{4}}e_2.$$

Thus, we have $\|y - z\| = a\|x\|$ and $\frac{\|x+y\|+\|x-z\|}{2} = \sqrt{2+a}.$

Theorem 3.3. *For a Banach space* X, $\frac{A_2(a, X)^2}{2} \le C_{NJ}(a, X)$ *for all* $a \in [0, \infty)$.

Lemma 3.4 [4]. *Let X be a Banach space. For* $0 \le a < 2$, *if* $C_{NJ}(a, X) = 2$, *then there exist sequences* $\{x_n\}, \{y_n\}, \{z_n\}$ *in* B_X *satisfying*:

(1) $\|x_n\|, \|y_n\|, \|z_n\| \to 1$;

(2) $\|x_n + y_n\|, \|x_n - z_n\| \to 2$;

(3) $\|y_n - z_n\| \le a\|x\|$ *for all n.*

Furthermore, the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ *can be chosen from* S_X.

Corollary 3.5. *For a Banach space* X, $A_2(a, X) = 2$, *if and only if* $C_{NJ}(a, X) = 2$ *for all* $a \in [0, 2]$.

Proof. If $C_{NJ}(a, X) = 2$, then by lemma, there exist sequences $\{x_n\}, \{y_n\}$, and $\{z_n\}$ in S_X satisfying $\|x_n + y_n\|, \|x_n - z_n\| \to 2$, and $\|y_n - z_n\| \le a$ for all n. Thus, $A_2(a, X) = 2$. The other direction is an easy consequence of proposition.

Corollary 3.6. *For a Banach space* X, $J(a, X) = 2$, *if and only if* $A_2(a, X) = 2$.

Corollary 3.7. *Let X be a Banach space. If* $J(1, X) < 2$ *or* $A_2(a, X) < 2$, *then X has uniform normal structure.*

Proposition 3.1. *For* $0 \le a \le b$, $A_2(b, X) + \frac{a}{2} \le A_2(a, X) + \frac{b}{2}$. *In particular,* $A_2(\cdot, X)$ *is continuous on* $[0, \infty)$.

Proof. Let $\varepsilon > 0$. There exist $x, y, z \in B_X$ such that $\|y - z\| = b_1\|x\|$, and $A_2(b, X) - \varepsilon \le \frac{\|x + y\| + \|x - z\|}{2}$. b_1 can be chosen so that $a < b_1$. Otherwise, the assertion is obviously true. We can choose $z_1, y_1 \in B_X$ such that $\|y - y_1\|, \|z - z_1\| \le \frac{b - a}{2}$, and $\|y_1 - z_1\| \le a\|x\|$. Then, we have

$$A_2(b, X) - \varepsilon \le \frac{\|x + y\| + \|x - z\|}{2}$$

$$\leq \frac{\|x+y_1\|+\|y-y_1\|+\|x-z_1\|+\|z-z_1\|}{2}$$

$$\leq \frac{\|x+y_1\|+\|x-z_1\|}{2}+\frac{b-a}{2}$$

$$\leq A_2(a,\ X)+\frac{b-a}{2}.$$

To finish the proof, we let $\varepsilon \to 0$.

Lemma 3.8. *Uniformly nonsquare Banach spaces are super-reflexive.*

Corollary 3.9. $A_2(a,\ X) = A_2(a,\ \widetilde{X})$.

Proof. Clearly, $A_2(a, X) \leq A_2\ (a, \widetilde{X})$. To show $A_2(a, X) \geq A_2(a, \widetilde{X})$, let $\delta > 0$, $\alpha \in [0, a]$ and suppose $\widetilde{x}, \widetilde{y}, \widetilde{z} \in \widetilde{X}$ and $\|\widetilde{y}-\widetilde{z}\| = \alpha\|\widetilde{x}\|$. If $\widetilde{x} = 0$, then $\frac{\|\widetilde{y}\|+\|\widetilde{z}\|}{2} = \|\widetilde{y}\| \leq 1 \leq A_2(a, X)$. If $\widetilde{x} \neq 0$, choose $\varepsilon > 0$ such that $\varepsilon < \delta\|\widetilde{x}\|$. Since

$$c := \frac{\|\widetilde{x}+\widetilde{y}\|+\|\widetilde{x}-\widetilde{z}\|}{2} = \lim_{U} \frac{\|x_n+y_n\|+\|x_n-z_n\|}{2} := \lim_{U} c_n,$$

the set $\{n \in N : |c_n - c| < \delta$ and $\|y_n - z_n\| \leq \alpha\|x_n\| + \varepsilon < (a+\delta)\|x_n\|\}$ belongs to U.

In particular,

$$c < \frac{\|x_n+y_n\|+\|x_n-z_n\|}{2} + \delta \leq A_2(a+\delta,\ X)+\delta \text{ for some } n.$$

Then, the inequality $A_2(a,\ X) \geq A_2(a,\ \widetilde{X})$ follows from the arbitrariness of δ and the continuity of $A_2(\cdot\ ,\ X)$.

Lemma 3.10. *Let X be a Banach space without weak normal structure, then for any $0 < \varepsilon < 1$ and each $0 \leq t \leq 1$, there exist $x_1 \in S_X$, $x_2, x_3 \in tS_X$ satisfying*:

(1) $x_2 - x_3 = ax_1$ *with* $|a-t| < \varepsilon$;

(2) $\|x_1 + x_2\| > (1+t) - 3\varepsilon$, $\|x_1 + (-x_3)\| > (1+t) - 3\varepsilon$.

Theorem 3.11. *Let X be a Banach space. If* $A_2(a, X) < 1 + a$, *for some* $a \in [0, 1]$, *then X has uniform normal structure.*

Proof. It suffices to show that these conditions imply X has normal structure. For the case $A_2(a, X) < 1 + a$, $a \in [0, 1]$, and Remark 3.1, X is uniformly nonsquare and so in turn is reflexive. Thus normal structure and weak normal structure coincide, it suffices to prove that X has weak normal structure. By the continuity of $A_2(\cdot, X)$, $A_2(a', X) < 1 + a$, for some $a' > a$. Choose $m \in N$ such that $a + \frac{1}{m} \leq a'$. Suppose X does not have weak normal structure, by lemma, there exist $x_n \in S_X$, $y_n, z_n \in aS_X$ such that, for each $n \in N$, $y_n - z_n = a_n x_n$ with $|a_n - a| < \frac{1}{n+m}$, $\|x_n + y_n\| > (1 + a) - \frac{3}{n+m}$, $\|x_n - z_n\| > (1 + t) - \frac{3}{n+m}$. $\|y_n - z_n\| = a_n < a + \frac{1}{n+m} \leq a'$ and $\liminf_{n\to\infty}\|x_n + y_n\| \geq 1 + a$, and $\liminf_{n\to\infty}\|x_n - z_n\| \geq 1 + a$. Thus,

$$1 + a \leq \liminf_{n\to\infty} \frac{\|x_n + y_n\| + \|x_n - z_n\|}{2} \leq A_2(a', X) < 1 + a.$$

This contradiction shows that X must have weak normal structure.

Corollary 3.12. *Let X be a Banach space. If* $A_2(1, X) < 2$, *then X has uniform normal structure.*

Theorem 3.13. *Let X be a Banach space,* $\varepsilon \in [0, 2]$, *and* $\beta \geq 0$. *If* $A_2(a, X) \leq \frac{2 + |\varepsilon - \beta|}{2}$, *then* $\delta_X(\varepsilon) \geq 0$.

Proof. Suppose $\delta_X(\varepsilon) = 0$, there exist $x_n, y_n \in S_X$, such that $\|x_n - y_n\| = \varepsilon$ for all $n \in N$, and $\liminf_{n\to\infty}\|x_n + y_n\| = 2$. Put $z_n = y_n - \beta x_n$. Then, for each $n \in N$, $y_n - z_n = \beta x_n$, $\|z_n\| = \|y_n - \beta x_n\| \leq 1 + \beta$, and $\|x_n - z_n\| \geq |\,\|x_n - y_n\| - \|\beta x_n\|\,| = |\varepsilon - \beta|$. Thus,

$$\frac{2 + |\varepsilon - \beta|}{2} \leq \liminf_{n\to\infty} \frac{\|x_n + y_n\| + \|x_n - z_n\|}{2} < \frac{2 + |\varepsilon - \beta|}{2}.$$

We obtain a contradiction.

Corollary 3.14. *If* $A_2(0, X) < 1 + \frac{\varepsilon}{2}$, *then* $\delta_X(\varepsilon) \geq 0$.

Corollary 3.15. *If* $A_2(\cdot, X)$ *is concave and* $A_2(a, X) < \frac{\sqrt{3} + 1 + (3 - \sqrt{3})a}{2}$ *for some* $a \in [0, 1]$, *then X has uniform normal structure.*

Proof. If $A_2(1, X) < 2$, we are done by Corollary 3.5. Let $A_2(1, X) = 2$ and suppose that X does not have uniform normal structure. Therefore, $A_2(0, X) \geq \frac{1 + \sqrt{3}}{2}$. By the concavity of $A_2(\cdot, X)$, we have, for all $a \in [0, 1]$

$$A_2(a, X) \geq (1 - a)A_2(0, X) + aA_2(1, X) \geq \frac{\sqrt{3} + 1 + (3 - \sqrt{3})a}{2},$$

a contradiction.

Example 3.16 ($l_\infty - l_1$norm)**.** Let $X = R^2$ be equipped with the norm defined by

$$\|x\| = \begin{cases} \|x\|_\infty & \text{if} \quad x_1x_2 \geq 0, \\ \|x\|_1 & \text{if} \quad x_1x_2 \leq 0. \end{cases}$$

Take $x = (1, 1)$, $y = (0, 1)$ and $z = (-1, 0)$. Then, we have $y - z = (1, 1) = x$ and $\|x + y\| = \|(1, 2)\|_\infty = 2$, $\|x - z\| = \|(2, 1)\|_\infty = 2$. So,

$$2 = (2 + 2)/2 = \frac{\|x + y\| + \|x - z\|}{2} \leq A_2(1, X) \leq 2.$$

Hence, $A_2(1, X) = 2$.

References

[1] M. Baronti, E. Casini and P. L. Papini, Triangles inscribed in a semicircle, in Minkowski planes, and in normed spaces, J. Math. Anal. Appl. 252 (2000), 124-146.

[2] M. M. Day, Some characterizations of inner product spaces, Trans. Amer. Math. Soc. 62 (1947), 320-337.

[3] S. Dhompongsa, P. Piraisangjun and S. Saejung, Generalized Jordan-von Neumann constants and uniform normal structure, Bull. Austral. Math. Soc. 67 (2003), 225-240.

[4] S. Dhompongsa, A. Kaewkhao and S. Tasena, On a generalized James constant, J. Math. Anal. Appl. 285 (2003), 419-435.

[5] T. Figiel, On the moduli of convexity and smoothness, Studia Math. 56 (1976), 121-155.

[6] J. Gao and K. S. Lau, On two classes of Banach spaces with uniform normal structure, Studia Math. 99 (1991), 41-56.

[7] J. Gao, Modulus of convexity in Banach spaces, J. Appl. Math. Lett. 16 (2003), 273-278.

[8] J. Garcia-Falset and B. Sims, Property (*M*) and the weak fixed point property, Proc. Amer. Math. Soc. 125 (1997), 2891-2896.

[9] J. Garcia-Falset, The fixed point property in Banach spaces with NUS-property, J. Math. Anal. Appl. 215 (1997), 532-542.

[10] B. Sims, A class of spaces with weak normal structure, Bull. Austral. Math. Soc. 49 (1994), 523-528.

[11] C. Yang and F. Wang, On a generality modulus of convexity, Journal of Henan Normal University 3(1) (2006), 55.

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